# A Note on the Heterogeneous Choice Model

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The heterogeneous choice model (HCM) has been proposed as an extension of the standard logit and probit models (Williams, 2009). In this note, I show that in an important special case, this model is just another way to specify an interaction effect.

For developing the argument I refer to a logit model. Let $Y$ denote a binary variable. The logit model makes the distribution of $Y$ dependent on values of explanatory variables:

$$\Pr(Y = 1 \,|\, X = x, Z = z) = L(\alpha + x\,\beta_x + z\,\beta_z) \tag{1}$$

where $X$ is a vector of explanatory variables with corresponding parameter vector $\beta_x$, $Z$ is a further explanatory variable with parameter $\beta_z$, $\alpha$ is a constant, and $L(x) := \exp(x)/(1 + \exp(x))$ is the standard logistic distribution function.

In (1), $L$ is not used as a distribution function for a corresponding random variable but as a function for linking covariates to the distribution of $Y$, that is, a binomial distribution characterized by a single parameter. The HCM starts from a different model that relates to a latent variable, say $Y^*$, which is connected with $Y$ through $Y = 1 \iff Y^* > 0$:

$$Y^* = \alpha + x\,\beta_x + z\,\beta_z + \epsilon \tag{2}$$

Assuming that $\epsilon$ has a standard logistic distribution, (1) can be derived from (2). However, in model (2) there is now a newly created random variable, $\epsilon$, and one can refer to its variance. While this variance cannot be estimated with data on $Y$, $X$ and $Z$, one nevertheless can set up a model which makes this variance dependent on some of the covariates. This is the HCM which, for example, can be specified as

$$Y^* = \alpha + x\,\beta_x + z\,\beta_z + \epsilon\,\exp(z\,\gamma) \tag{3}$$

where it is now assumed that the residual variance depends on the value of $Z$ ($\exp(.)$ is used to guarantee a positive value). Dividing by $\exp(z\,\gamma)$, one gets the corresponding logit formulation

$$\Pr(Y = 1 \,|\, X = x, Z = z) = L(\alpha/\exp(z\gamma) + x\,\beta_x/\exp(z\gamma) + z\,\beta_z/\exp(z\gamma)) \tag{4}$$

I now consider the special case where $Z$ is a binary variable. Instead of (4), one can consider a logit model with an interaction term $x\,z$ (being a vector of interaction terms if $x$ is a vector):

$$\Pr(Y = 1 \,|\, X = x, Z = z) = L(\alpha^* + x\,\beta^*_x + z\,\beta^*_z + x\,z\,\beta^*_{xz}) \tag{5}$$

Both models are equivalent, meaning that they entail the same conditional probabilities. This can be achieved by setting

$$\begin{aligned} &\alpha^* = \alpha,\ \beta_x^* = \beta_x \\ &\beta_z^* = \alpha\,(1/\exp(\gamma) - 1) + \beta_z/\exp(\gamma) \\ &\beta_{xz}^* = \beta_x\,(1/\exp(\gamma) - 1) \end{aligned} \tag{6}$$

In fact, these relationships are implicitly satisfied when estimating the models with maximum likelihood. They also entail that one cannot add the $\exp(z\gamma)$ term to a logit model which already includes an interaction term.

How to interpret the equivalence of (4) and (5) depends on the intended analysis. I suppose that one is interested in investigating how a binary variable, $Y$, depends on another binary variable (representing, e.g., two groups). The interest concerns the dependence of the probabilities $\Pr(Y = 1 \mid X = x, Z = z)$ on values of $Z$, or on values of $X$ conditional on values of $Z$. One should then start from model (5) which includes a parameter, $\beta_{xz}^*$, for a possible interaction between $X$ and $Z$ (in addition to interactions entailed by the nonlinearity of the logit link function).

Given a significant interaction parameter, one might ask whether the HCM suggests a different interpretation. The answer depends on the understanding of the HCM. I first assume that the HCM is taken as a model for the binary variable $Y$ as formulated in (4). Then, if $X$ and $Z$ interact in model (5), the same is true for model (4). Of course, both models provide different parameter values, but this is just a consequence of a different parameterization of the same model. For the research interest mentioned above, it is only important that both parameterizations entail identical conditional probabilities.

Now I assume that the HCM is taken as a model for the latent variable $Y^*$ as formulated in (3). Referring to the conditional expectation of $Y^*$, that is

$$\mathrm{E}(Y^* \mid x, z) = \alpha + x\beta_x + z\beta_z$$

suggests the conclusion that the effect of $X$ on $Y^*$ is independent of $Z$. However, the interest eventually concerns the effect of $X$ on $Y$, not on $Y^*$, and the effect of $X$ on $Y$ not only depends on the expectation of $Y^*$, but also on its variance. Consequently, if the variance of $Y^*$ depends on $Z$, as assumed by the HCM, one immediately gets an interaction between $X$ and $Z$ (as already shown by the equivalence of (4) and (5)).

## References

Williams, R. (2009). Using Heterogeneous Choice Models to Compare Logit and Probit Coefficients Across Groups. *Sociological Methods & Research* 37, 531–559.